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Perturbations of the signum-Gordon model

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Abstract

We investigate a perturbation of a scalar field model (called here the signum-Gordon model) with the potential $V(f) = |f|$. The perturbation generalizes the signum-Gordon model to the signum-Klein-Gordon model, i.e. to the case $V(f) = |f| - \frac{1}{2}\lambda f^2$, where λ is a small parameter. Such a generalization breaks the scaling symmetry of the signum-Gordon model. In this paper we concentrate on solutions for self-similar initial data. Such data are particularly useful for identification of the effects caused by the term that breaks the scaling symmetry. We have found that the behaviour of the solutions is quite interesting—they escape and return periodically to the self-similar initial data.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The present paper refers to recently investigated scalar field models with V-shaped potential [1–3]. Such potentials have a common feature—left- and right-hand derivatives are different from zero at the minimum. Mentioned models have a well-justified physical origin despite the fact that they seem to be a little bit exotic from a mathematical viewpoint. Moreover, some physical systems described by scalar field models with V-shaped potentials are easy to build (e.g. chain of pendulums impacting on a rectilinear bar). Unfortunately, such models have a very unpleasant mathematical feature—a typical solution consists of many (sometimes infinitely many) partial solutions. The partial solutions are matched at some points. The matching procedure is mostly onerous. This is probably the reason why literature is poor in results for field theoretic models with V-shaped potential.

It turns out that the behaviour of the field close to minimum strongly depends on a ‘shape’ of the potential [4]. In particular, for V-shaped potential a field approaches exactly its vacuum value at finite distance (a parabolic approach). This fact has a profound significance—kinks have no exponential tails! Such kinks are called *compactons* because their supports are compact. The compactons considered in our models are topological, so they are qualitatively different from, e.g., well-known compactons in the modified KdV model [5–7]. Recently, the

topological compactons have been also obtained in models with nonstandard kinetic terms as so-called k -defects [8]. It is important to note that there are other (nontopological) compact solutions in the signum-Gordon (s-G) model [9]. Because of their properties they are called *oscillons*.

The second characteristic property of the mentioned models is a scaling symmetry (see [2]). This symmetry means that if a function $f(x, t)$ is a solution of a field equation, then new function defined as $f_\nu(x, t) = \nu^2 f(x/\nu, t/\nu)$ is a solution as well. ν is here a positive constant. A presence of the scaling symmetry in the model suggests existence of solutions that are invariant with respect to the scaling transformation (so-called self-similar solutions). Such solutions have been obtained in the s-G model. A complete list of solutions for the self-similar initial data is presented in [10]. For models with the potential $V(f) = af\Theta(-f) + bf\Theta(f)$, where a, b are constant parameters and Θ is the well-known Heaviside step function, the scaling symmetry is exact whereas for most models with V-shaped potential the symmetry is only approximated. Note that the s-G model can be obtained as a particular case, i.e. by setting $a = -1$ and $b = 1$. In a group of models with symmetric V-shaped potential, the s-G model is the simplest one. In this paper we study just symmetric potentials.

The aim of the present work is the analysis of the perturbed s-G model, where for simplicity reasons the specific perturbation is chosen in the simplest, nontrivial form. Namely, we add the quadratic term. In spite of its simplicity such a generalization of the s-G model allows us to face several important problems. The first one is breaking of the scaling symmetry. Among the physical systems there are fewer of them with an exact scaling symmetry. There are always fluctuations in a typical physical system that interacts with its environment. The fluctuations modify an effective potential and break the exact scaling symmetry. In this physical context, it is clear that investigation of the perturbed field theoretic models with V-shaped potentials is an important issue. It allows for better understanding of dynamics of compactons in the systems with the broken scaling symmetry. In our paper we analyse the perturbed potential $V(f) = |f| - \frac{1}{2}\lambda f^2$, where λ is a small parameter, i.e. $|\lambda| \ll 1$. The second important problem, which is in general very difficult for systems with non-differentiable potentials, is a stability analysis of solutions. Our investigations are some kind of structural stability analysis. Such analysis is important for compact kinks as well as compact oscillons.

However, the models with V-shaped potentials are interesting from the mathematical point of view, they have also some properties that allow us to think about possible applications to condense matter physics and cosmology as well. In the cosmological context, the most interesting seems to be the fact that for models with potentials sharp at its minima the terms that come from a gradient of the potential dominate the field dynamics close to the minimum. For instance, in the s-G model the term $\frac{dV}{df} = \text{sign} f$ remains finite arbitrary close to the minimum. This is in total opposition to the behaviour of, e.g., ϕ^4 theory, where the gradient of the potential vanishes close to the minimum. Because of this, small perturbations propagate easily within the topological compactons or other nontopological field configurations like, e.g. mentioned oscillons, whereas outside of them the propagation encounters on resistance. Moreover, an absence of linear perturbations around the V-shaped minimum is a basic feature of our models. It entails automatically that the linear perturbations can propagate only at a defect background. This effect, characteristic for models with V-shaped potentials, is similar to the behaviour of k -fields that play a prominent role in cosmology (see [11]).

In our calculation, we concentrate on differences between solutions in the s-G model and solutions in the signum-Klein-Gordon (s-K-G) model (the perturbed model). Applying the same initial data for solutions in both these models we can analyse the differences between their solutions as a pure effect caused by the term $\frac{1}{2}\lambda f^2$. In the case when the initial data are

self-similar (parabolic) the solutions in the s-G model have especially simple form. We apply the self-similar data just for this reason.

Our paper is organized as follows. In section 2, we introduce the signum-Klein-Gordon (s-K-G) model and give a general method of calculation of partial solutions that can be obtained directly from the self-similar initial data. Unfortunately, they are insufficient to construct a solution valid for each x and $t > t_0$, where t_0 is an initial moment. Section 3 is devoted to a study of a solution for a specific self-similar initial data. Focusing on a specific initial data enables us to calculate all partial solutions that (when matched together) cover the whole range of variable x . In the last section, we summarize our results and emphasize effects that stem from the term $\frac{1}{2}\lambda f^2$.

2. Initial problem for the generalized model

2.1. The signum-Klein-Gordon model and its partial solutions

The s-K-G model for the scalar field $f(x, t)$ in (1+1) dimensions has the Lagrangian

$$L = \frac{1}{2}(\partial_t f)^2 - \frac{1}{2}(\partial_x f)^2 - V(f), \tag{1}$$

where the potential $V(f)$ is given by the formula

$$V(f) = |f| - \frac{1}{2}\lambda f^2. \tag{2}$$

Euler-Lagrange equation that corresponds to Lagrangian (1) takes the following form:

$$(\partial_t^2 - \partial_x^2)f + \text{sign} f - \lambda f = 0. \tag{3}$$

The sign of parameter λ has a crucial meaning for the behaviour of the field $f(x, t)$. The potentials $V(f)$ for negative and positive values of parameter λ are qualitatively different [3]. The case $\lambda = 0$ gives the s-G model which has been discussed in our previous papers (see, e.g., [1, 2, 10]). In this paper we are interested in the case $\lambda < 0$, because for $\lambda > 0$ the potential $V(f)$ is not bounded from below. Nevertheless, a perturbative method presented in the following subsection involves both cases of sign λ . In order to distinguish between different kind of solutions, we use symbol f for solutions in the s-K-G model ($\lambda \neq 0$) and symbol ϕ for solutions in the s-G model. The partial self-similar solutions are given by the formula

$$\phi_k(x, t) = \frac{(-1)^k (x - v_{k-1}t)(x - v_k t)}{2 v_{k-1}v_k - 1}, \tag{4}$$

where $x \in [v_{k-1}t, v_k t]$ and $k = 1, 2, \dots$. The parameters v_k are velocities of zeros of polynomials. They are determined from matching conditions. For more details see [10]. The partial solutions obey the relation $\text{sign}\phi_k = (-1)^{k+1}$. By analogy, we define partial solutions in the model with $\lambda \neq 0$. They obey the equation

$$(\partial_t^2 - \partial_x^2)f_k(x, t) - (-1)^k - \lambda f_k(x, t) = 0, \tag{5}$$

where $\text{sign}f_k = (-1)^{k+1}$. For $|\lambda| \ll 1$ the potential $V(f) = |f| - \frac{1}{2}\lambda f^2$ can be interpreted as a perturbed potential $V(\phi) = |\phi|$. In this case, we say that the exact scaling symmetry is violated or that the generalized model has an approximate scaling symmetry when $|f| \gg \lambda f^2$.

2.2. Self-similar initial data and partial solutions

This paper is devoted to investigations which are the effects caused by the term λf in the s-K-G equation. It can be achieved by comparison of two solutions for the same initial data: the first one that is a solution in the s-G model and the second one that comes from the s-K-G model.

The differences between them are a direct consequence of the term that breaks the scaling symmetry. From practical reasons we investigate some characteristic points of solutions, i.e. trajectories of its zeros.

It turns out that explicit formulae for the solutions are not always available—this problem strongly depends on initial data. It has been shown, see [10], that solutions in the s-G model for the self-similar (parabolic) initial data are given by explicit formulae. For this reason, the self-similar initial data are more useful for our purposes than other, more general, initial data. In fact, any self-similar solution ϕ at the moment $t = t_0$ is suitable for our purposes and can be applied as an initial data. Therefore, we assume the following initial data for the partial solutions:

$$f_k(x, t_0) = \phi_k(x, t_0), \quad \partial_t f_k(x, t)|_{t=t_0} = \partial_t \phi_k(x, t)|_{t=t_0}. \quad (6)$$

2.3. The perturbative method

The method presented in the current subsection allows us to obtain the partial solutions $f_k(x, t)$ directly from the initial data (6). We call them *the partial solutions of the first kind*. It turns out that such partial solutions are insufficient. A complete solution $f(x, t)$ consists of some other partial solutions as well. This inconvenience appears also for, e.g., the s-G equation in the case when initial data have the form of piecewise smooth functions matched up at some points. In most cases, such matching points are origins of new partial solutions. At the initial moment $t = t_0$ mentioned partial solutions are shrunk to single points but for $t > t_0$ their supports expand (*the partial solutions of the second kind*). The s-K-G model has analogical partial solutions (the first and the second kind)—we discuss their properties in the further part of this paper. The partial solutions discussed in this paragraph are the first kind ones. They cover whole range of the axis x at $t = t_0$ (they obey (6) where ϕ_k have this property) and their supports shrink for $t > t_0$.

Let us assume that the solutions $f_k(x, t)$, which depend on parameter λ , are represented in the form of a power series

$$f_k(x, t) = \sum_{n=0}^{\infty} f_{kn}(x, t)\lambda^n, \quad (7)$$

where $\lambda \ll 1$. After plugging series (7) into equation (5) we get a set of equations

$$\begin{cases} (\partial_t^2 - \partial_x^2) f_{k0}(x, t) = (-1)^k & n = 0, \\ (\partial_t^2 - \partial_x^2) f_{kn}(x, t) = f_{k(n-1)}(x, t) & n = 1, 2, 3, \dots \end{cases} \quad (8)$$

Each of them takes the form of wave equation with a source. We can integrate them using new variables $\xi = \frac{1}{2}(x + t)$, $\eta = \frac{1}{2}(x - t)$. The result of integration in the original variables reads

$$f_{k0}(x, t) = F_0(x + t) + G_0(x - t) - \frac{(-1)^k}{4}(x^2 - t^2), \quad (9)$$

$$f_{kn}(x, t) = F_n(x + t) + G_n(x - t) - \int_0^{\frac{x+t}{2}} d\alpha \int_0^{\frac{x-t}{2}} d\beta f_{k(n-1)}(\alpha + \beta, \alpha - \beta), \quad (10)$$

where $F(x + t)$ and $G(x - t)$ are arbitrary functions. They can be calculated from the following initial conditions for the partial solutions:

$$f_{k0}(x, t_0) = \phi_k(x, t_0), \quad \partial_t f_{k0}(x, t)|_{t=t_0} = \partial_t \phi_k(x, t)|_{t=t_0}, \quad (11)$$

$$f_{kn}(x, t_0) = 0, \quad \partial_t f_{kn}(x, t)|_{t=t_0} = 0. \quad (12)$$

Conditions (11) and (12) stem from the initial conditions (6). The solution (9) obeys the first of equations (8) (i.e. the s-G equation) and the self-similar initial data (11), so it coincides with ϕ_k ,

$$f_{k0}(x, t) = \phi_k(x, t). \tag{13}$$

A direct calculation confirms this result. It means that the self-similar solutions (13) are zero-order approximation for the solutions (7). In order to get higher-order approximations we have to find the functions $F_n(x + t)$ and $G_n(x - t)$. Differentiating first of equations (12) with respect to x , combining with the second one and shifting arguments, we obtain equations

$$D_+ f_{kn}(x, t)|_{x=s-t_0, t=t_0} = 0, \quad D_- f_{kn}(x, t)|_{x=w+t_0, t=t_0} = 0, \tag{14}$$

where $D_{\pm} \equiv \frac{1}{2}(\partial_x \pm \partial_t)$. Equations (14) can be rewritten in the form

$$F'_n(s) = D_+ \int_0^{\frac{s+t}{2}} d\alpha \int_0^{\frac{s-t}{2}} d\beta f_{kn-1}(\alpha + \beta, \alpha - \beta) \Big|_{x=s-t_0, t=t_0}, \tag{15}$$

$$G'_n(w) = D_- \int_0^{\frac{s+t}{2}} d\alpha \int_0^{\frac{s-t}{2}} d\beta f_{kn-1}(\alpha + \beta, \alpha - \beta) \Big|_{x=w+t_0, t=t_0}, \tag{16}$$

where formula (10) has been applied. The sum of constants that comes from integration of the expressions (15) and (16) is fixed by the first of conditions (12).

In the first step we calculate the function $f_{k1}(x, t)$ from formula (10). Then we can continue the procedure in order to obtain $f_{k2}(x, t)$. In principle, this procedure can be repeated infinitely many times giving expressions for all functions $f_{kn}(x, t)$. In fact, we are able to obtain only a few functions $f_{kn}(x, t)$ because the calculations quickly became too complicated. Fortunately, functions $f_{kn}(x, t)$ obtained for several, the lowest values of n enable us to guess a general formula for arbitrary n . This formula has the form

$$f_{kn}(x, t) = (-1)^k \frac{(t - t_0)^{2n} A_n x^2 + B_n x + C_n}{(2n + 2)! v_{k-1} v_k - 1}, \tag{17}$$

where

$$\begin{aligned} A_n &= (2n + 1)(n + 1), \\ B_n &= -(v_{k-1} + v_k)(n + 1)(t + 2nt_0), \\ C_n &= (n + v_{k-1} v_k)t^2 + n(1 + (2n + 1)v_{k-1} v_k)t_0^2. \end{aligned}$$

One can check that formula (17), which was originally found for $n = 1, 2, \dots$, is also true for $n = 0$ —in this case it gives (13).

It turns out, and this is a big surprise, that the series (7) can be summed up giving as a result

$$\begin{aligned} f_k(x, t) &= \frac{(-1)^k}{2(v_{k-1} v_k - 1)} \left[M \cosh(\rho(t - t_0)) + N \frac{\sinh(\rho(t - t_0))}{\rho} \right] \\ &\quad + \frac{(-1)^k}{\rho^2} [\cosh(\rho(t - t_0)) - 1], \end{aligned} \tag{18}$$

for $\lambda \equiv \rho^2 > 0$, and

$$\begin{aligned} f_k(x, t) &= \frac{(-1)^k}{2(v_{k-1} v_k - 1)} \left[M \cos(\sigma(t - t_0)) + N \frac{\sin(\sigma(t - t_0))}{\sigma} \right] \\ &\quad + \frac{(-1)^k}{\sigma^2} [1 - \cos(\sigma(t - t_0))], \end{aligned} \tag{19}$$

for $\lambda \equiv -\sigma^2 < 0$. The coefficients M and N read

$$M \equiv (x - v_{k-1}t_0)(x - v_k t_0), \quad N \equiv t - t_0 + 2v_{k-1}v_k t_0 - (v_{k-1} + v_k)x.$$

The partial solutions f_k satisfy the relation $\text{sign } f_k = (-1)^{k+1}$.

It is worth emphasizing that the partial solutions f_k in the model with the explicitly broken scaling symmetry, still have a quadratic dependence on variable x . What changes is the time dependence. This is the first important result obtained with the help of the perturbative method.

Knowing this result we can, of course, propose *a posteriori* a proper Ansatz

$$f_k(x, t) = a(t)x^2 + b(t)x + c(t). \tag{20}$$

Plugging the Ansatz (20) into equation (5) we get the set of ordinary differential equations for the coefficients $a(t)$, $b(t)$ and $c(t)$

$$\frac{d^2a}{dt^2} - \lambda a = 0, \quad \frac{d^2b}{dt^2} - \lambda b = 0, \quad \frac{d^2c}{dt^2} - \lambda c = 2a + (-1)^k.$$

The constants that come from integration of these equations are fixed by condition (6). Solving these equations we recover formulae (18) and (19) depending on sign of the parameter λ .

3. The solution for a specific initial data

In this section, we present a solution in the s-K-G model for a specific self-similar initial data. We concentrate on the case when the perturbative parameter λ is small $|\lambda| \ll 1$ and negative $\lambda = -\sigma^2$. The main effort is focused on the partial solutions of the second kind. They can be obtained as the solutions of boundary problem. The boundary conditions are given by the partial solutions of the first kind at points of contact of their supports and supports of the partial solutions of the second kind. The most serious obstacle in the s-K-G model is that we do not know a general formula for the partial solutions. For instance, in the s-G model such formula consists of two arbitrary functions and terms $\pm t^2/2$, $\pm x^2/2$ or their combinations. For this reason, we restrict our study to a specific self-similar initial data and construct an approximated solution. Nevertheless, it turns out that investigation of such a specified case gives valuable information as well. One of the most interesting results presented in the current section is a discovery of periodicity in time for such solution in the s-K-G model.

3.1. The positive partial solution $f_+(x, t)$

Among the self-similar initial data, the simplest one reads

$$\phi(x, 0) = \frac{1}{4}x^2\Theta(x), \quad \partial_t \phi(x, t)|_{t=0} = 0, \tag{21}$$

where $\Theta(x)$ is the Heaviside step function. The self-similar solution $\phi(x, t)$ in the s-G model for data (21) is consisted of two partial solutions matched up at the light cone $x = t$ (see also [10]). The solution $\phi(x, t)$ has a very simple form

$$\phi(x, t) = \frac{1}{4}(x^2 - t^2)\Theta(x - t). \tag{22}$$

This formula corresponds to (4) for $k = 1$, $v_0 = -1$ and $v_1 = 1$. The snapshot of the solution $\phi(x, t)$ is presented in figure 1.

In the further part of this paper we concentrate on the solution $f(x, t)$ in the s-K-G model. The positive partial solution $f_+(x, t)$ obeys the equation

$$(\partial_t^2 - \partial_x^2)f_+ + \sigma^2 f_+ + 1 = 0, \tag{23}$$

and the initial conditions

$$f_+(x, 0) = \frac{1}{4}x^2, \quad \partial_t f_+(x, t)|_{t=0} = 0.$$

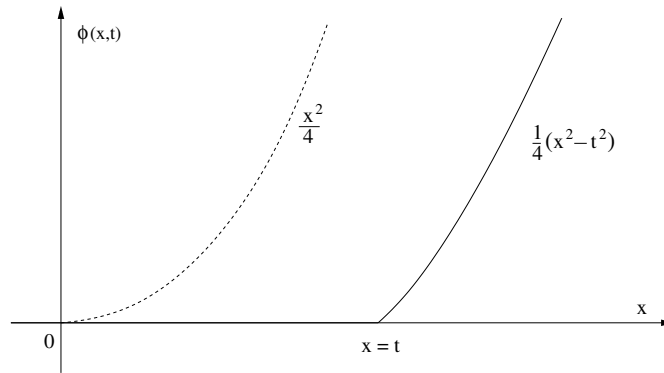


Figure 1. The self-similar solution for initial data (21). The dashed line represents an initial configuration of the field ϕ .

It takes the form

$$f_+(x, t) = \frac{1}{4} \cos(\sigma t)x^2 + \frac{t}{4\sigma} \sin(\sigma t) - \frac{1}{\sigma^2}(1 - \cos(\sigma t)). \tag{24}$$

The partial solution (24) can be obtained directly from (19) for $k = 1$ and $t_0 = 0$. The formula (24) holds for $x \geq x_1(t)$. The trajectory of zero $x_1(t)$,

$$x_1(t) = \frac{2}{\sigma} \sqrt{\frac{1}{\cos(\sigma t)} - \frac{\sigma t}{4} \tan(\sigma t)} - 1, \tag{25}$$

is the solution of the equation $f_+(x_1, t) = 0$. Except for the point $t = 0$, the function $x_1(t)$ obeys inequality $x_1(t) > t$, which means that zero of $f_+(x, t)$ moves faster than its counterpart ($x(t) = t$) in the s-G model. Moreover, we can see that the velocity of zero $x_1(t)$ depends on variable t . Let us remind that velocities v_k of zeros of the self-similar solutions are constant (see formula (4)). It means that the term λf in equation (3) is responsible for non-monotonous expansion or contraction of the supports of the partial solutions.

A series expansion of the formula (25) for small t ,

$$x_1(t) = t + \frac{1}{4}t^3\sigma^2 + \frac{103}{1440}t^5\sigma^4 + O(t^7), \tag{26}$$

gives valuable information as well. We can see from (26) that the zero $x_1(t)$ moves with the acceleration $\ddot{x}_1(t)$ which is proportional to σ^2 provided that $t \ll 1$. This observation has a practical meaning because it enables us to calculate the parameter λ from experimental data. The curve $x_1(t)$ is presented in figure 2. We can see a very good agreement between the analytical calculation and the numerical data. The function (25) goes to infinity for $t \rightarrow t^* \equiv \frac{\pi}{2\sigma^2}$. It means that the solution $f_+(x, t)$ is valid for $t < t^*$. In our numerical calculation ($\sigma^2 = 0.1$) the characteristic time $t^* \approx 4.9673$. The leading behaviour of $x_1(t)$ close to t^* is given by the first term of the expansion

$$x_1(t) = \frac{1}{2\sigma^{3/2}} \sqrt{\frac{16 - 2\pi}{t^* - t}} - \frac{3}{\sigma^{1/2}} \sqrt{\frac{t^* - t}{16 - 2\pi}} + O(\sigma^{1/2}).$$

3.2. The negative partial solution $f_-(x, t)$ —some general remarks

In this and two further paragraphs we present the partial solution of the second kind $f_-(x, t)$. Our solution is only approximated and holds for times not longer than $t \approx 2$. Such a partial

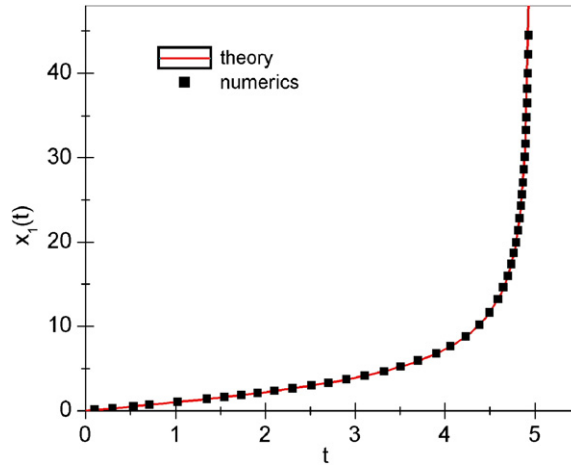


Figure 2. The trajectory $x_1(t)$ for $\lambda = -0.1$.

solution can be obtained as the solution of boundary problem because at the initial time $t = 0$ its support is a single point and it is located at $x = 0$. For later times $t > 0$ the support expands to infinite size. This behaviour has been observed in our numerical calculation and it is suggested by the fact that $x_1(t)$ tends to infinity for $t \rightarrow \frac{\pi}{2\sigma}$. The solution $f_-(x, t)$ has to obey the following boundary conditions:

$$f_-(x_1, t) = 0, \quad \partial_x f_-(x, t)|_{x=x_1} = \partial_x f_+(x, t)|_{x=x_1}, \quad (27)$$

$$f_-(x_0, t) = 0, \quad \partial_x f_-(x, t)|_{x=x_0} = 0. \quad (28)$$

Conditions (27) and (28) are derived from the field equation (3) and they mean that the partial solutions are matched so that the solution $f(x, t)$ is smooth at $x_1(t)$ and $x_0(t)$. At $x_0(t)$ the partial solution $f_-(x, t)$ is matched with the trivial partial solution $f_0(x, t) = 0$. Whereas the point $x_1(t)$ is given by the formula (25), the second zero of $f(x, t)$, i.e $x_0(t)$, is not known yet. In section 3.4, we show how to obtain an approximated formula for $x_0(t)$.

It turns out that the partial solution that obeys (27) does not obey (28) and vice versa. In order to get rid of this inconvenience we split the solution $f_-(x, t)$ into two pieces $f_-^L(x, t)$ and $f_-^R(x, t)$. They are matched at the light cone $x = t$. Such split is sufficient to obtain $f_-(x, t)$ that obeys (27) and (28) simultaneously. The solution $f(x, t)$ consists of the following partial solutions:

$$f(x, t) = \begin{cases} 0 & \text{for } x \leq x_0(t), \\ f_-^L(x, t) & \text{for } x_0(t) \leq x \leq t, \\ f_-^R(x, t) & \text{for } t \leq x \leq x_1(t), \\ f_+(x, t) & \text{for } x \geq x_1(t), \end{cases}$$

where $t < \frac{\pi}{2\sigma}$. The snapshot of $f(x, t)$ is depicted in figure 3—compare it to the solution in figure 1.

3.3. The partial solution $f_-^R(x, t)$

It has been already mentioned at the beginning of section 3 that a general formula for the partial solutions in the s-K-G model is not known. This is the most serious obstacle in

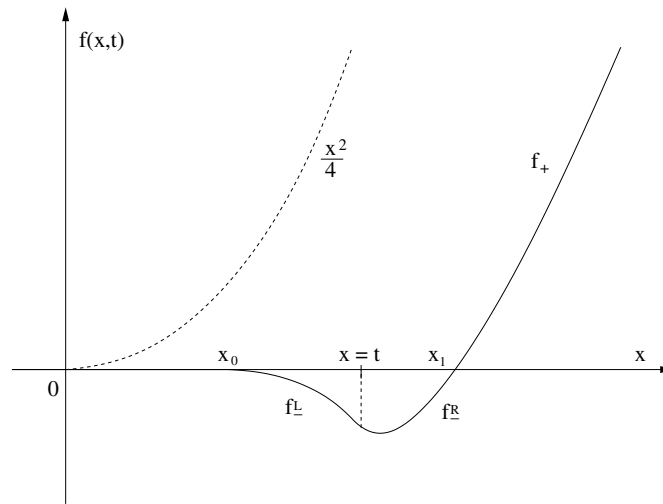


Figure 3. The solution $f(x, t)$ for initial data (21) and times $t < \frac{\pi}{2\sigma}$. The dashed line represents an initial configuration of the field f .

our investigations. Therefore, we search for the approximated partial solution $f_-(x, t)$. The approximation of equation (3) is obtained by replacing the term $\sigma^2 f(x, t)$ by the term $\sigma^2 \phi(x, t)$, which gives us

$$(\partial_t^2 - \partial_x^2)f_-(x, t) + \sigma^2 \phi(x, t) - 1 = 0. \tag{29}$$

Such a modification is valid only for small times. The partial solution $f_-^R(x, t)$ of equation (29) (an approximate solution of equation (3) for $\lambda = -\sigma^2$) at $t \leq x \leq x_1(t)$ has the form

$$f_-^R(x, t) = F_R(x+t) + G_R(x-t) - \frac{1}{4}(x^2 - t^2) + \frac{\sigma^2}{64}(x^2 - t^2)^2. \tag{30}$$

In accordance with (27), this solution is matched to $f_+(x, t)$ at $x = x_1(t)$. It turns out that exact formulae for $F_R(x+t)$ and $G_R(x-t)$ cannot be achieved because we need the inverse functions of $x_1(t) \pm t$, where $x_1(t)$ is given by (25). Nevertheless, we can expand the expressions $x_1(t) \pm t$ in power series and then invert these series up to an arbitrary term. This is why we concentrate on series expansions of the partial solutions. In our further calculations we use the perturbation parameter σ as an expansion parameter. The partial solutions represented by finite series (i.e. approximated partial solutions) obey the matching conditions (27) and (28) up to some range of σ . In order to find this range we start from series expansion of the formula (25) for $\sigma \ll 1$, i.e.,

$$x_1(t) = t + \frac{1}{4}t^3\sigma^2 + \frac{103}{1440}t^5\sigma^4 + O(\sigma^6). \tag{31}$$

Note that the expansion (31) has the same form as the expansion (26) for small times t . A leading term of the expression $x_1(t) - t$ is proportional to σ^2 . The powers of the expression $x_1(t) - t$ appear in (27) because the partial solution $f_-^R(x, t)$, which is given by formula (30), includes terms proportional to $x - t$ and $\sigma^2(x - t)^2$. In order to take into consideration contributions from all terms in (30), especially from $\sigma^2(x - t)^2$, we need terms proportional to σ^6 at least. This is, naively, an accuracy of $f_-^R(x, t)$. The real accuracy is lower. The direct calculations allow us to obtain $f_-^R(x, t)$ only up to terms proportional to σ^4 because solutions

of (27), i.e., F'_R and G'_R have such accuracy (see formulae below). The prime ' stands for derivatives with respect to whole arguments of F_R and G_R .

In the first step we expand formulae (24), (30) and their derivatives with respect to variable x at $x = x_1(t)$, where $x_1(t)$ is given by the formula (26). The result has the following form:

$$f_+(x_1(t), t) = O(\sigma^6),$$

$$\partial_x f_+(x, t)|_{x=x_1} = \frac{1}{2}t - \frac{1}{8}t^3\sigma^2 - \frac{17}{2880}t^5\sigma^4 + O(\sigma^6),$$

$$f_-^R(x_1, t) = F_R(x_1 + t) + G_R(x_1 - t) - \frac{1}{8}t^4\sigma^2 - \frac{37}{720}t^6\sigma^4 + O(\sigma^6),$$

$$\partial_x f_-^R(x, t)|_{x=x_1} = F'_R(x_1 + t) + G'_R(x_1 - t) - \frac{1}{2}t - \frac{1}{8}t^3\sigma^2 - \frac{13}{2880}t^5\sigma^4 + O(\sigma^6).$$

Plugging three last formulae into conditions (27) we obtain two equations that contain F_R , G_R , F'_R and G'_R . Then, we differentiate the first of these equations, i.e. $f_-(x_1(t), t) = 0$ with respect to variable t . In the next step we solve obtained equations with respect to F'_R and G'_R , which gives

$$F'_R(x_1 + t) = -\frac{1}{8}t^3\sigma^2 - \frac{71}{2880}t^5\sigma^4 + O(\sigma^6), \tag{32}$$

$$G'_R(x_1 - t) = t + \frac{1}{8}t^3\sigma^2 + \frac{67}{2880}t^5\sigma^4 + O(\sigma^6). \tag{33}$$

Equation (32) can be integrated with the help of new variable $s = x_1(t) + t$, where $x_1(t)$ is given by series (31). In the inverse series

$$t(s) = \sum_{k=0}^N b_k s^{2k+1} \sigma^{2k}, \tag{34}$$

only terms up to $N = 1$ are significant to ensure the given accuracy. Coefficients b_0 and b_1 have the following numerical values:

$$b_0 = \frac{1}{2}, \quad b_1 = -\frac{1}{64}.$$

Consequently, the approximate formula for $F_R(s)$ takes the form

$$F_R(s) = -\frac{1}{256}s^4\sigma^2 + \frac{1}{8640}s^6\sigma^4 + O(\sigma^6). \tag{35}$$

In the similar way we compute $G_R(w)$, where $w = x_1(t) - t$. The inverse series is given by the formula

$$t(w) = \sum_{k=0}^N c_k w^{\frac{2k+1}{3}} \sigma^{\frac{2k-2}{3}}. \tag{36}$$

In this case we have to compute coefficient c_k up to $N = 7$. We will not present here their numerical values. Function $G_R(w)$ takes the form

$$G_R(w) = \sum_{k=0}^7 g_k w^{\frac{2k+4}{3}} \sigma^{\frac{2k-2}{3}},$$

where the coefficients g_k have the following approximated numerical values:

$$\begin{aligned} g_0 &= 1.1900, & g_1 &= 0.0593, & g_2 &= -0.0299, & g_3 &= -0.0429, \\ g_4 &= 0.0037, & g_5 &= -0.0134, & g_6 &= 0.0022, & g_7 &= 0.0002. \end{aligned}$$

It is important to note that the term $k = 0$ in $G_R(w)$ has a singular dependence on σ , i.e. it is proportional to $\sigma^{-2/3}$. Such behaviour is caused by the fact that term proportional to σ^0 , linear in t , is cancelled in the definition of the variable w which is proportional to $\sigma^2 t^3$ for $t \ll 1$. In fact, the singular term $\sigma^{-2/3}$ appears already in the series (36).

3.4. The partial solution $f_-^L(x, t)$

For $x \leq t$ the self-similar solution (22) is equal to zero, therefore equation (29) takes the form

$$(\partial_t^2 - \partial_x^2)f_-^L(x, t) - 1 = 0. \tag{37}$$

The general solution of (37) is of the form

$$f_-^L(x, t) = F_L(x + t) + G_L(x - t) - \frac{1}{4}(x^2 - t^2). \tag{38}$$

The arbitrary functions F_L, G_L and unknown function $x_0(t)$ can be obtained after imposing the following matching conditions:

$$f_-^L(t, t) = f_-^R(t, t), \quad f_-^L(x_0, t) = 0, \quad \partial_x f_-^L(x, t)|_{x=x_0} = 0. \tag{39}$$

The first of conditions (39) gives equality of values of the partial solutions f_-^R and f_-^L at the light cone $x = t$. We do not require equality of spatial derivatives of these partial solutions but it turns out that equality of values entails equality of derivatives as well. The matching condition at $x = t$ gives an equality $F_L(2t) + G_L(0) = F_R(2t)$, which allows us to obtain the function F_L . Without loosing of generality we can fix $G_L(0) = 0$, because $G_L(0)$ is cancelled in the combination $F_L(x + t) + G_L(x - t)$. Last two matching conditions (39) allow us to obtain derivatives of functions F_L and G_L . Differentiating the second condition in (39) with respect to variable t and combining with the third one we obtain

$$D_+ f_-^L(x, t)|_{x=s-t(s), t=t(s)} = 0, \quad D_- f_-^L(x, t)|_{x=w+t(w), t=t(w)} = 0, \tag{40}$$

where $s = x_0(t) + t, w = x_0(t) - t$ and $D_\pm \equiv \frac{1}{2}(\partial_x \pm \partial_t)$. The solutions of (40) take the form

$$F'_L(s) = \frac{1}{4}(s - 2t(s)), \quad G'_L(w) = \frac{1}{4}(w + 2t(w)). \tag{41}$$

In order to obtain $x_0(t)$, we use the equality $F'_L(s) = F'_R(s)$, which gives the equation

$$\frac{\sigma^4}{45}s^5 - \frac{\sigma^2}{2}s^3 - 8s + 16t = 0.$$

A solution of this equation $s(t)$ can be obtained in the series form. Finally, it gives $x_0(t)$ in the form

$$x_0(t) = t - \frac{1}{2}t^3\sigma^2 + \frac{167}{360}t^5\sigma^4 + O(\sigma^6). \tag{42}$$

A leading term of deceleration of zero x_0 is proportional to σ^2 for $t \ll 1$. Figure 4 depicts the trajectory of $x_0(t)$. The analytical curve which is given by the first three terms in formula (42) is a good approximation for a numerical trajectory if t is not greater than $t \approx 2$. It gives the limitation for validity of the approximated partial solution $f_-(x, t)$.

The explicit formula for $x_0(t)$ enables us to obtain the function G_L and the solution $f_-^L(x, t)$. Function $x_0(t)$, given by (42), is known up to $O(\sigma^4)$ (including this term) so $t(w)$ can be obtained up to $O(\sigma^0)$. The second formula in (41) gives

$$G_L(w) = -\frac{3}{8} \left(\frac{2}{\sigma^2} \right)^{1/3} w^{4/3} - \frac{4}{135}w^2 + O(\sigma^{2/3}). \tag{43}$$

Finally, we obtain the partial solution $f_-^L(x, t)$ of the form

$$f_-^L(x, t) = -\frac{1}{540}(151x - 119t)(x - t) - \frac{\sigma^2}{256}(x + t)^2 + \frac{\sigma^4}{8640}(x + t)^4 - \frac{3}{8} \left(\frac{2}{\sigma^2} \right)^{1/3} (x - t)^{4/3}. \tag{44}$$

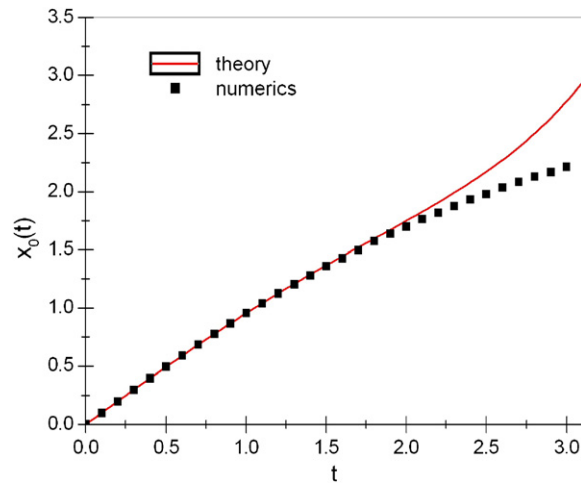


Figure 4. Trajectory of $x_0(t)$.

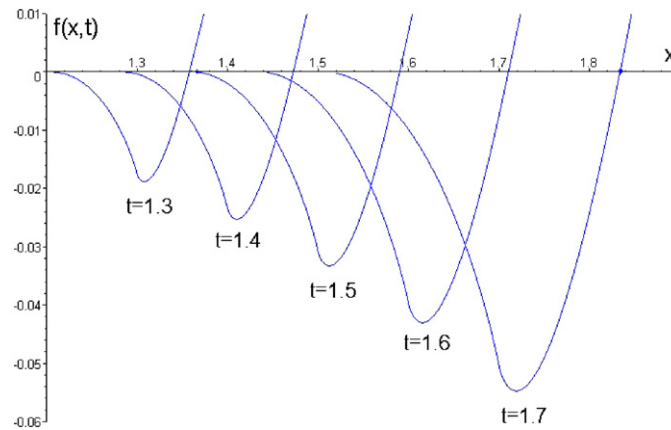


Figure 5. Evolution of approximated analytical solution $f(x, t)$ for $\sigma^2 = 0.1$.

We note that the singular term $\sim \sigma^{-2/3}$ is present in the formulae (43) and (44). There is nothing unexpected in this fact because for small times the function $x_0(t) - t$ has similar behaviour to the function $x_1(t) - t$ from the previous paragraph. Some snapshots of the solution $f(x, t)$ for first stage of evolution are presented in figures 5 and 6. There is a very good agreement between numerical and analytical solutions until $t \approx 2$.

3.5. Behaviour for later times $t > t^*$

In the current section we study a numerical solution for the s-K-G model with $\lambda = -0.1$. The solution at $t = 0$ obeys following conditions:

$$f(x, 0) = \frac{1}{4}x^2\Theta(x), \quad \partial_t f(x, t)|_{t=0} = 0.$$

We focus on times $t > t^*$, where $t^* \approx 4.9673$. The trajectories of zeros of $f(x, t)$ up to $t = 25$ are depicted in figure 7.

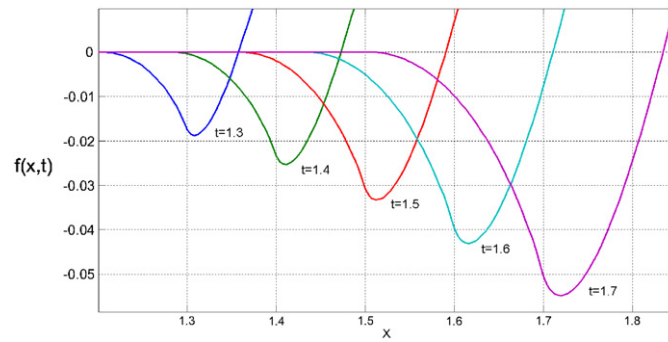


Figure 6. Numerical solution $f(x, t)$ for $\sigma^2 = 0.1$.

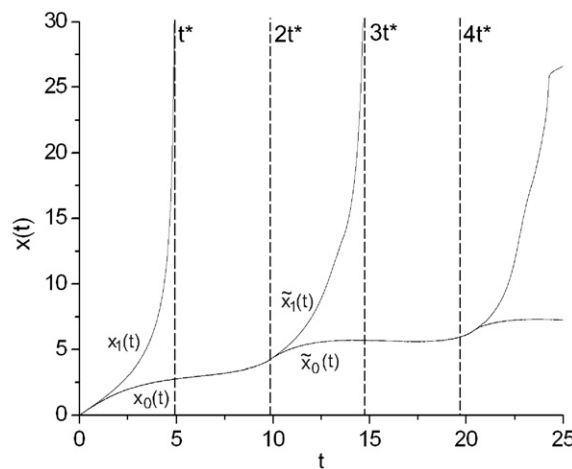


Figure 7. Trajectories of zeros.

Zero $x_0(t)$ decelerates up to time $t \approx t^*$, then it accelerates up to $t \approx 2t^*$ when it reaches the velocity $v = 1$. At this moment a very interesting phenomenon occurs. Zero $x_0(t)$ splits into a pair of zeros $\tilde{x}_0(t)$ and $\tilde{x}_1(t)$ that move similarly to $x_0(t)$ and $x_1(t)$. Numerical values of $\tilde{x}_1(t)$ increase rapidly for $t \rightarrow 3t^*-$, which suggests that function $\tilde{x}_1(t)$ has a vertical asymptote at $t = 3t^*$. This phenomenon seems to have an almost periodic character.

One can show by an elementary calculation (we skip the proof) that the point x_0 , at which the trivial solution ($f = 0$) and a nonzero solution are matched so that the first right-hand spatial derivative of f at x_0 is equal to zero, cannot move with velocity $v \geq 1$. The solution $f(x, t)$, that consists of the partial solutions matched at the points that move with the velocity $v \neq 1$, has to be smooth (i.e first derivative is continuous) at the matching points. We can conclude that an accelerating zero, that is a matching point of the trivial partial solution $f = 0$ and other nonzero partial solution, cannot move faster than $v = 1$ without changing its character. This change means that the first spatial derivative of the field $f(x, t)$ is nonzero at this point. It is possible provided that an additional zero, that moves with velocity $v < 1$, appears. The segment of the x -axis between those two zeros widens. Such segment is a support of a partial solution of the second kind. At the moment when x_0 reaches the velocity

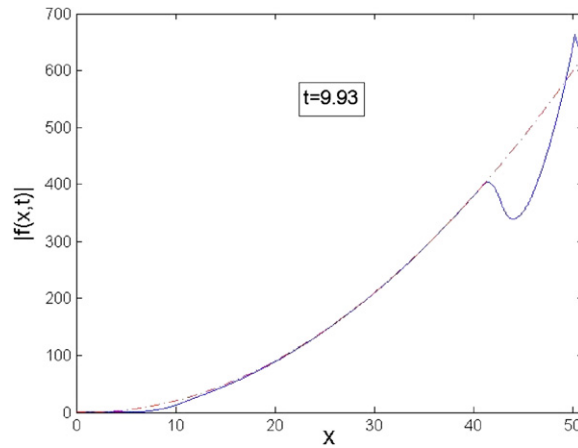


Figure 8. Absolute value of the solution $f(x, t)$ at $t = 2t^*$ —the solid line, and the parabola of the self-similar initial data—dashed line. A discrepancy at the right-hand side is a pure numerical effect caused by finite size of the grid.

$v = 1$ the solution $f(x, t)$ returns with good approximation to its self-similar initial data (45) (see figure 8).

4. Summary

A study of perturbed field theoretic models with V-shaped potential and the scaling symmetry can give some knowledge that are the properties of more general field models with potentials that are sharp at its minima. One of the simplest example of such models is the s-G model with perturbation that introduces an additional linear term to the field equation, which gives the s-K-G model. We have found that such a perturbation is responsible for some new effects. For the same initial data the differences between solutions in the s-G and the s-K-G models are caused by this linear term.

The first observation comes from analysis of trajectories of zeros. The zeros of the self-similar solutions in the s-G model move with constant velocities, whereas velocity of zeros of solutions in the s-K-G model is not monotonous. Moreover, their accelerations and decelerations also depend on time. Nevertheless, for small times $t \ll 1$ both the acceleration and the deceleration of zeros are proportional to the perturbation parameter λ . This fact can be useful as a phenomenological criterion that allows us to calculate the parameter λ from experimental data.

We can also point out the second qualitative difference between these models. There are partial solutions in the s-K-G model that supports expand from zero to infinite size within a finite time. It is possible provided that the other partial solutions disappear simultaneously. For instance, the partial solution $\phi_1 = \frac{1}{4}(x^2 - t^2)$ at $[t, \infty)$ in the s-G model disappears when $t \rightarrow \infty$, whereas the partial solution f_+ in the s-K-G model disappears when $t \rightarrow \frac{\pi}{2\sigma}$.

One of the most interesting results that have been obtained from approximated formulae for partial solutions $f_-^R(x, t)$ and $f_-^L(x, t)$ is an observation that these partial solutions contain singular terms proportional to $\lambda^{-1/3}$. It suggests that (apart from technical obstacles) the partial solution $f_-(x, t)$ cannot be represented in the series form in the similar way to $f_+(x, t)$.

The last observation is mainly numerical. The trajectories of zeros of the solution $f(x, t)$ are almost periodic. We have proposed here a hypothesis that the period has the value $2t^* = \frac{\pi}{\sigma}$,

where t^* has been obtained from analytical calculations—it is the characteristic time for which the trajectory of zero $x_1(t)$ has a vertical asymptote. Our hypothesis agrees quite well with the numerical data. The periodicity for longer than investigated times is an open question. In our study the solutions $f(x, 0)$ and $f(x, 2t^*)$ are very similar which means that the solution $f(x, t)$ returns to the self-similar initial data even though the scaling symmetry is broken.

An important and open question is the behaviour of solutions of the s-K-G model for other self-similar initial data or more general initial data. In the group of general initial data the most interesting are these for which an initial field configuration has a finite energy (the energy of the self-similar solutions has an infinite value). Finally, there are, of course, a variety of perturbations of the potential $V(\phi) = |\phi|$ that can be studied, nevertheless, it is clear that for most of the analytical results it can be obtained only in approximation.

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